[1]

裏が n 回，表が $10-n$ 回

出る確率 P_n は

$$P_n = {}_{10}C_n\left(\frac{1}{2}\right)^n {}_{10-n}C_{10-n}\left(\frac{1}{2}\right)^{10-n}$$

$$= {}_{10}C_n\left(\frac{1}{2}\right)^{10}$$

ここで

$$10^7 \leqq 8^n \times 3^{10-n} < 10^8$$

より

$$7 \leqq \log_{10}(8^n \times 3^{10-n}) < 8$$

ここで

$$\log_{10}(8^n \times 3^{10-n})$$

$$= \log_{10}2^{3n} + \log_{10}3^{10-n}$$

$$= 3n\log_{10}2 + (10-n)\log_{10}3$$

$$= 3n \times 0.3010 + (10-n) \times 0.4771$$

$$= 0.9030n + 4.771 - 0.4771n$$

$$= 0.4259n + 4.771$$

$$7 \leqq 0.4259n + 4.771 < 8$$

$$2.229 \leqq 0.4259n < 3.229$$

$$5 < n < 7.6$$

$n = 0, 1, 2, \cdots, 10$ より

$$n = 6, 7$$

よって，求める確率は

$$P_6 + P_7 = {}_{10}C_6\left(\frac{1}{2}\right)^{10} + {}_{10}C_7\left(\frac{1}{2}\right)^{10}$$

$$= {}_{11}C_7\left(\frac{1}{2}\right)^{10}$$

$$= \frac{11 \times 10 \times \cancel{9}^{3} \times \cancel{8}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1} \times \frac{1}{2^{10}}$$

$$= \frac{165}{512}$$

[2]

$f(x) = x^3 - kx^2 + kx + 1$ とおくと

$f'(x) = 3x^2 - 2kx + k$

これが極値をもつためには

$f'(x) = 0$ の判別式 $D > 0$

となるとよい.

$D/4 = k^2 - 3k > 0$

$k(k-3) > 0$

よって $k < 0,\ 3 < k$ ——— ①

また, $f'(x) = 0$ を解くと,

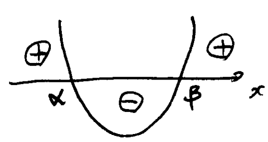
$$x = \frac{k \pm \sqrt{k^2 - 3k}}{3}$$

であり,

$$\alpha = \frac{k - \sqrt{k^2-3k}}{3},\quad \beta = \frac{k + \sqrt{k^2-3k}}{3}$$

とおくと.

$f'(x)$ の符号!



よって $\begin{cases} \text{極大値} & f(\alpha) \\ \text{極小値} & f(\beta) \end{cases}$

であり.

$f(\alpha) - f(\beta) = 4|k|^3$ より

$[f(x)]_{\beta}^{\alpha} = 4|k|^3$

$\int_{\beta}^{\alpha} f'(x)dx = 4|k|^3$

$\int_{\beta}^{\alpha} 3(x-\alpha)(x-\beta)dx = 4|k|^3$

$-\frac{1}{2}(\alpha - \beta)^3 = 4|k|^3$

(ここで

$\alpha - \beta = -\frac{2}{3}\sqrt{k^2-3k}$ より)

$\frac{4}{27}(k^2-3k)^{\frac{3}{2}} = 4|k|^3$

$(k^2-3k)^{\frac{3}{2}} = 27|k|^3$

$(k^2-3k)^{\frac{3}{2}} = |3k|^3$

よって

$k^2 - 3k = 9k^2$

$8k^2 + 3k = 0$

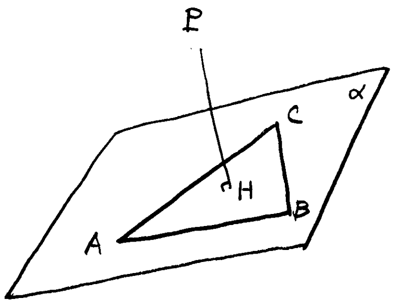
$k(8k+3) = 0$

$k = 0,\ -\frac{3}{8}$

① より

$$\underline{\underline{k = -\frac{3}{8}}}$$

[3]



(1) $\overrightarrow{OH} = \overrightarrow{OA} + (s\overrightarrow{AB} + t\overrightarrow{AC})$

とでき

$$\left(\begin{aligned}
\overrightarrow{AB} &= -\overrightarrow{OA} + \overrightarrow{OB} \\
&= (-1,-2,-3) + (3,2,3) \\
&= (2,0,0) \\
\overrightarrow{AC} &= -\overrightarrow{OA} + \overrightarrow{OC} \\
&= (-1,-2,-3) + (4,5,6) \\
&= (3,3,3)
\end{aligned}\right)$$

$\overrightarrow{OH} = (1,2,3) + s(2,0,0) + t(3,3,3)$

$\quad = (2s+3t+1,\ 3t+2,\ 3t+3)$

$\overrightarrow{PH} = -\overrightarrow{OP} + \overrightarrow{OH}$

$\quad = (-6,-p,-q) + (2s+3t+1,\ 3t+2,\ 3t+3)$

$\quad = (2s+3t-5,\ -p+3t+2,\ -q+3t+3)$

$\overrightarrow{PH} \perp$ 平面α より

$\overrightarrow{PH} \perp \overrightarrow{AB}$ かつ $\overrightarrow{PH} \perp \overrightarrow{AC}$

$\Leftrightarrow \underbrace{\overrightarrow{PH}\cdot\overrightarrow{AB} = 0}_{①}$ かつ $\underbrace{\overrightarrow{PH}\cdot\overrightarrow{AC} = 0}_{②}$

① より

$2(2s+3t-5) = 0$

$2s+3t = 5$ ——— ①′

② より

$3(2s+3t-5-p+3t+2-q+3t+3) = 0$

$2s+9t-p-q = 0$

$2s+9t = p+q$ ——— ②′

①′ かつ ②′ より

$s = \dfrac{-p-q+15}{4}$, $t = \dfrac{p+q-5}{6}$

このとき

$\overrightarrow{PH} = \left(0,\ \dfrac{-p+q-1}{2},\ \dfrac{p-q+1}{2}\right)$

よって

$|\overrightarrow{PH}| = \sqrt{\left(\dfrac{-p+q-1}{2}\right)^2 + \left(\dfrac{p-q+1}{2}\right)^2}$

$\quad = \dfrac{1}{2}\sqrt{2(p-q+1)^2}$

$\quad = \dfrac{\sqrt{2}}{2}|p-q+1|$

(2) $S(\triangle ABC) = \dfrac{1}{2}\sqrt{|\overrightarrow{AB}|^2|\overrightarrow{AC}|^2 - (\overrightarrow{AB}\cdot\overrightarrow{AC})^2}$

$$\left(\begin{aligned}
&\text{ここで} \\
&|\overrightarrow{AB}| = 2,\quad |\overrightarrow{AC}| = \sqrt{9+9+9} = 3\sqrt{3} \\
&\overrightarrow{AB}\cdot\overrightarrow{AC} = 6
\end{aligned}\right)$$

$S(\triangle ABC) = \dfrac{1}{2}\sqrt{4\times27-36}$

$\quad = 3\sqrt{2}$

よって

$V(ABCP) = \dfrac{1}{3}\times S(\triangle ABC)\times|\overrightarrow{PH}|$

$\quad = \dfrac{1}{3}\times3\sqrt{2}\times\dfrac{\sqrt{2}}{2}|p-q+1|$

$\quad = |p-q+1|$

点Pは $(p-9)^2+(q-7)^2=1$

を満たすので

$\begin{cases} p-9 = \cos\theta \\ q-7 = \sin\theta \end{cases} \Leftrightarrow \begin{cases} p = 9+\cos\theta \\ q = 7+\sin\theta \end{cases}$

なる θ $(0 \leqq \theta < 2\pi)$ が存在する.

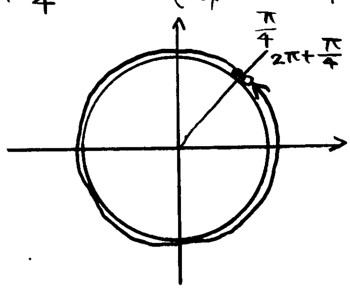
このとき

$p-q+1 = (9+\cos\theta) - (7+\sin\theta) + 1$

$\quad = \cos\theta - \sin\theta + 3$

$\quad = \sqrt{2}\cos\left(\theta + \dfrac{\pi}{4}\right) + 3$

$\theta + \frac{\pi}{4}$ の図!! $\left(\frac{\pi}{4} \leqq \theta + \frac{\pi}{4} < 2\pi + \frac{\pi}{4}\right)$



$$-1 \leqq \cos\left(\theta + \frac{\pi}{4}\right) \leqq 1$$

$$-\sqrt{2} + 3 \leqq \sqrt{2}\cos\left(\theta + \frac{\pi}{4}\right) + 3 \leqq \sqrt{2} + 3$$

$$3 - \sqrt{2} \leqq p - q + 1 \leqq 3 + \sqrt{2}$$

$$3 - \sqrt{2} \leqq |p - q + 1| \leqq 3 + \sqrt{2}$$

$$3 - \sqrt{2} \leqq V(ABCP) \leqq 3 + \sqrt{2}.$$

よって

$$\begin{cases} \mathrm{Max}\, V(ABCP) = 3 + \sqrt{2} \\ \mathrm{min}\, V(ABCP) = 3 - \sqrt{2} \end{cases}$$

〔4〕 は 文理共通